Exam Linear Algebra

(WILAICL-09)

Monday 16 June 2014, 9:00 - 12:00

During this exam, the textbook, lecture notes and a simple (non-programmable) calculator may be consulted. Use of any other electronic device is prohibited, including (smart-)phones, e-readers and mp3-players.
Motivate and explain your calculations and answers. Write explicitly the steps when performing a Gaussian elimination and show how a determinant is calculated. Write clearly and use separate paper for scratch calculations. Please do not seal the envelope.
Good luck!

Note: There are 6 exercises on 3 pages.

On each paper sheet, clearly write your name, S-number and major (CS, AI, HIO).

Bonus: [10]

1. Given the vectors in $\mathbb{R}^3$: $\mathbf{a} = (\beta \;\; -1 \;\; 2)^T$, $\mathbf{b} = (1 \;\; -1 \;\; \alpha)^T$ and $\mathbf{c} = (\alpha \;\; -4 \;\; 1)^T$, where α and β are constants to be determined.

   (a) [5] Assume β=1. For which value(s) of α are **a**, **b** and **c** linearly dependent? For each α, express vector **a** as a linear combination of vectors **b** and **c**.

   (b) [4] For which values of α and β provide **a**, **b** and **c** an orthogonal basis for $\mathbb{R}^3$?

   (c) [4] Assume $\alpha = 3$ and $\beta = -1$. Compute a vector **v** of unit length that is orthogonal to both **a** and **b**.

2. Given the set of linear equations

$$\begin{aligned} x + y + 7z &= -7 \\ 2x + 3y + 17z &= 11 \\ x + 2y + (\alpha^2+1)z &= 6\alpha \end{aligned}$$

   (a) [4] Assume $\alpha = -1$ and determine the solution by means of Gauss elimination of the augmented matrix. Indicate which row operations you performed.

   (b) [6] For which values of α are there 0, 1 or ∞-many solutions?

   (c) [4] Use Cramer's rule to compute the value of y in terms of α. Verify your answer to 2a) with $\alpha = -1$.

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3. Given the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ where

$$T\begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}2\\7\\\alpha-2\end{pmatrix}, \quad T\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}-1\\9\\2\alpha-4\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}1\\-2\\0\end{pmatrix} = \begin{pmatrix}-1\\-4\\3\end{pmatrix}$$

(a) [4] Derive the corresponding 3x3 matrix A of T.

(b) [3] Assume $\alpha = -3$. Is T surjective ("onto")? Motivate your answer.

(c) [3] Assume $\alpha = 0$. Is T bijective? Motivate your answer.

(d) [4] After transformation under A with $\alpha = 0$, the range of T is transformed further under

$$B = \begin{pmatrix}1 & -2 & -1\\0 & 1 & 3\end{pmatrix}$$

Compute the corresponding matrix C of this composite linear transformation S.

(e) [3] Is the transformation S injective ("one-to-one")? Motivate your answer.

4. Given the matrix A

$$A = \begin{pmatrix}4 & -3 & \alpha & 0 & 0\\7 & -5 & -\alpha & 0 & 0\\0 & 0 & 2 & 0 & 0\\0 & 0 & 0 & 3 & -4\\0 & 0 & 0 & 2 & -2\end{pmatrix}$$

(a) [4] Compute $\det(A)$ and $\det(A^{-1})$.

(b) [4] Compute A^2 for $\alpha = 0$.

(c) [6] Compute A^{-1} for any α.

5. A whirlwind acts on a football that rolls on a flat playing field.
Its motion over the field is described by the following set of differential equations

$$\begin{aligned} x_1'(t) &= 3x_1(t) + x_2(t) \\ x_2'(t) &= -2x_1(t) + x_2(t) \end{aligned}$$

(a) [5] Calculate the eigenvalues and eigenvectors of this whirlwind.

(b) [7] Determine the general real solution, given that $\mathbf{x}(0) = (3 \;\; -2)^T$.

(c) [4] Describe the trajectory of a ball on the playing field.
Is the whirlwind an attractor or a repeller for footballs?

6. Two planes V and W in 3D-space are described by the following normal equations:

$$\begin{aligned} V: &\quad x + 2y - z = \alpha \\ W: &\quad -3x + 8y - 3z = 5 \end{aligned}$$

(a) [3] Are V and W perpendicular to each other? Motivate your answer.

(b) [3] For $\alpha = 0$, provide 2 vectors that span the plane V.

(c) [5] For $\alpha = 1$, provide a vector representation of the plane V.

(d) [5] For $\alpha = 1$, give a vector representation of the intersection line ℓ of V and W.

Total : [100]